Synchronizing Automata and the Černý Conjecture

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Seminar on Automata Theory

DFA

Definition

A DFA is a 5-tupel $(Q, \Sigma, \delta, q_0, F)$

with

- Q a finite set of states
- Σ an alphabet
- $\delta : \mathcal{P}(Q) \times \Sigma^* \to Q$ the transition function
- $q_0 \in Q$ the starting state
- $F \subseteq Q$ a set of final states

Synchronising DFA

Definition

A DFA is a 3-tupel $(Q, \Sigma, \delta : \mathcal{P}(Q) \times \Sigma^* \to \mathcal{P}(Q))$.

Definition

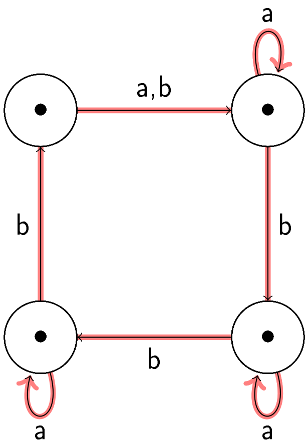
A word is *synchronising* with respect to an automaton $\mathcal{A} = (Q, \Sigma, \delta)$ if it leaves the automaton in a certain state, no matter which state the automaton started in.

w is synchronising in $\mathcal{A}$ $\Leftrightarrow$ For all $q \in Q$ it holds that $\delta(q, w) = q_0$

Definition

A *synchronising* DFA is a DFA that has a synchronising word.

Example



Extension

Observation

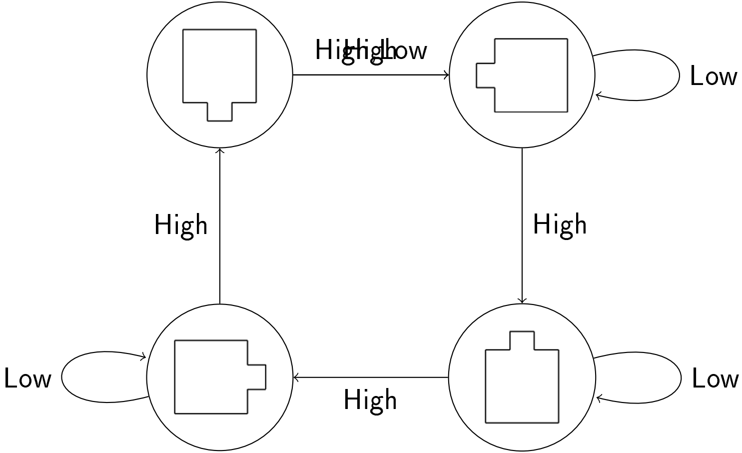
w is synchronising $\Rightarrow u \cdot w \cdot v$ is synchronising for all $u, v \in \Sigma^*$

Orientation on Conveyor Belts

- Problem: Orienting parts on conveyor belts [Ananichev and Volkov, 2004]

- Tools: Two types of barriers
  - High: Turns part by 90°
  - Low: Only turns part if the "nose" is down
- First solution: Sensors
  - High need for maintenance
  - High production costs
- Better solution: Synchronising automata

Orientation on Conveyor Belts



Example from [Ananichev and Volkov, 2004]

Biological Automata

- [Benenson et al., 2003] presents biological DFA
- "Soup of automata" containing $3 \cdot 10^{12}$ automata per μl
- Input: DNA molecules
- Possibility for highly parallel computing
- Problem: Bring automata back to starting state
- Solution: Build synchronising automata

Checking for synchronising words

- $L_{sync} := \{\mathcal{A} \mid \mathcal{A} \text{ is a synchronising DFA}\}$
- Decision problem: Input DFA $\mathcal{A}$, Output $\mathcal{A} \in L_{sync}$
- Idea: Keep track of states we can possibly be in

$\Rightarrow$ Power automaton

Checking for synchronising words

Input: A DFA $\mathcal{A} = (Q, \Sigma, \delta)$

1. Construct power automaton $\mathcal{P}(\mathcal{A}) = (\mathcal{P}(Q), \Sigma, \delta')$
   - $\delta' : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q), \delta'(P, a) := \{\delta(p, a) \mid p \in P\}$
2. Traverse DFA from $Q \in \mathcal{P}(Q)$ via BFS
3. If some state P with $|P| = 1$ is reached, return true, false otherwise

Output: *true* if the automaton is synchronising, *false* otherwise

Runtime

Construction	$\mathcal{O}(\lvert\mathcal{P}(Q)\rvert) = \mathcal{O}(2^{\lvert Q\rvert})$
BFS	$\mathcal{O}(\lvert\mathcal{P}(Q)\rvert) = \mathcal{O}(2^{\lvert Q\rvert})$
Sum	$\mathcal{O}(2 \cdot \lvert\mathcal{P}(Q)\rvert) = \mathcal{O}(2^{\lvert Q\rvert})$

Exponential runtime

Acceleration

Lemma

DFA $\mathcal{A}$ is synchronising $\Leftrightarrow$ *For each pair of states p, q, there is a word w such that $\delta(p, w) = \delta(q, w)$*

Proof.

"$\Rightarrow$" Choose synchronising word of $\mathcal{A}$ as w.
"$\Leftarrow$"

- Start in P
- Go from $P \subseteq Q$ to $P' \subseteq Q$ with $|P'| < |P|$
- Possible since there are $p, q \in P$ that can be unified
- Repeat until $|P'| = 1$
- $\Rightarrow$ Singleton state reachable from Q in power automaton
- $\Rightarrow$ $\mathcal{A}$ is synchronising

$\square$

Polynomial algorithm

- New idea: Check reachability of singleton state from all $P \in \mathcal{P}(Q)$ with $|P| = 2$

Input: DFA $\mathcal{A}$

1. Construct $\mathcal{P}^{[2]}(\mathcal{A}) := (Q_{pot}, \Sigma, \delta')$
   - $Q_{pot} \subseteq \mathcal{P}(Q)$, contains all states of cardinality 1 and 2
2. $W := \emptyset$
3. For each singleton $P \in Q_{pot}$
   1. Get the states R it is reachable from
   2. Set $W := W \cup (R \cap \{S \in Q_{pot} \mid |S| = 2\})$
4. If $R = \{P \in Q_{pot} \mid |P| = 2\}$, return *true*, else return *false*

Output: *true* if the automaton is synchronising, *false* otherwise

Complexity

Construction of automaton	States:	$\mathcal{O}(\frac{\lvert Q\rvert\lvert Q-1\rvert}{2}) = \mathcal{O}(\lvert Q\rvert^2)$
	Transitions:	$\mathcal{O}(\lvert Q\rvert^4)$
	Total:	$\mathcal{O}(\lvert Q\rvert^4)$
BFS for each singleton state	$\mathcal{O}(\lvert Q_{pot}\rvert \cdot \lvert Q_{pot}\rvert) = \mathcal{O}(\lvert Q\rvert^2)$	
Accumulated	$\mathcal{O}(\lvert Q\rvert^4)$	

No information about length of shortest synchronising word anymore

Length of synchronising words

$$L_{ShortResetWord} := \{(\mathcal{A}, l) \mid \text{DFA } \mathcal{A} \text{ has synchronising word of length } l\}$$

$$L_{ShortestResetWord} := \{(\mathcal{A}, l) \mid \text{ Minimal synchronising word of DFA } \mathcal{A} \text{ has length } l\}$$

NP-completeness

Lemma

$L_{ShortResetWord} =: L_{SRW}$ is NP-complete

Proof. (I)

$L_{SRW} \in$ NP: Guess w of length l nondeterministically, check in $\mathcal{O}(|Q| \cdot |w|)$
L_{SRW} NP-hard: Reduce from 3-SAT.

Idea

3-SAT		L_{SRW}
Given a formula,		Given an automaton,
does there exist	$\Leftrightarrow$	*does there exist*
an assignment that makes		a word that maps
all clauses	$\Leftrightarrow$	*all states*
true?		to the same state?

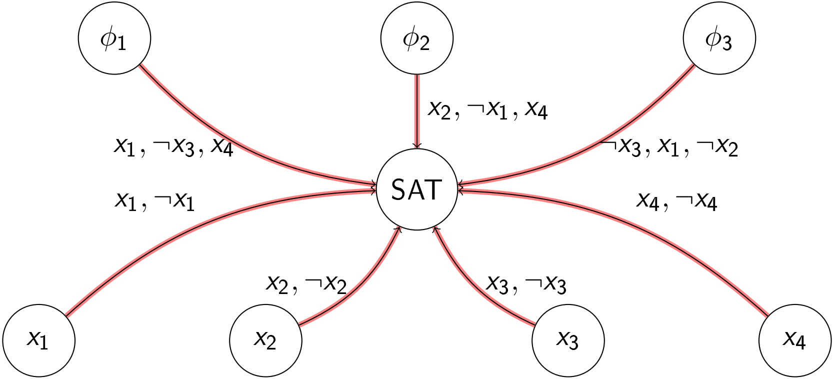
Reduction from 3-SAT

Input: Formula $\Phi = \bigwedge_{i=1}^{n} \phi_i$, $\phi_i = \bigvee_{j=1}^{3} x_{ij}$

1. $Var(\Phi) :=$ Variables in Φ, $Lit(\phi) :=$ Literals in ϕ
   - $\phi := (x_1, \neg x_1, x_2) \Rightarrow Var(\phi) = \{x_1, x_2\}, Lit(\phi) = \{x_1, \neg x_1, x_2\}$
2. $Q := \{\phi_i \mid 1 \leq i \leq n\} \cup Var(\Phi) \cup \{\text{SAT}\}$
3. $\Sigma := \{x, \neg x \mid x \in Var(\Phi)\}$
4. $\delta(q, a) := \begin{cases} \text{SAT}, & q \in Var(\Phi), a = q \text{ or } a = \neg q \\ q, & q \in Var(\Phi), a \neq q \text{ and } a \neq \neg q \end{cases}$

Output: Automaton $\mathcal{A} = (Q, \Sigma, \delta)$, $(\mathcal{A}, |Var(\Phi)|) \in L_{SRW} \Leftrightarrow \Phi \in 3-SAT$

$$\Phi = (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_1 \vee x_4) \wedge (\neg x_3 \vee x_1 \vee \neg x_2)$$
$$\phi_1 = (x_1 \vee \neg x_3 \vee x_4), \phi_2 = (x_2 \vee \neg x_1 \vee x_4), \phi_3 = (\neg x_3 \vee x_1 \vee \neg x_2)$$



$$\alpha(x_1) = 1 \ \alpha(x_2) = 1 \ \alpha(x_3) = 0 \ \alpha(x_4) = 0$$

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$\mathcal{O}(3n)$	$\mathcal{O}(n)$
Transitions	$\mathcal{O}(n) \cdot \mathcal{O}(n)$	$\mathcal{O}(n^2)$
Accumulated	$\mathcal{O}(n^2)$	

Correctness

$\Phi \in 3 - SAT \Rightarrow (\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$.

- $\Phi = \bigwedge_{i=1}^{n} \phi_i \in 3 - SAT$
- $\Rightarrow$ There is an assignment α making at least one literal in every clause true
- Pick $w := a_1 \dots a_n$ with $a_i = \begin{cases} x_i, & \alpha(x_i) = 1 \\ \neg x_i, & \text{else} \end{cases}$
  - w maps every state ϕ_i to SAT
  - w maps every literal x_i to SAT
  - $\Rightarrow w$ is synchronising
- $\Rightarrow \mathcal{A}$ has a synchronising word of length $|Var(\Phi)|$
- $\Rightarrow (\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$

$\square$

Correctness (cont.)

$(\mathcal{A}, |Var(\Phi)|) \in L_{SRW} \Rightarrow \Phi \in 3-SAT$.

- $(\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$
- $\Rightarrow \mathcal{A}$ has a synchronising word w of length $|Var(\Phi)|$
  - w does not describe a valid assignment $\Rightarrow$ w is not synchronising
  - $\Rightarrow$ w describes a valid assignment

  - Implied assignment α does not satisfy Φ
  - $\Rightarrow$ one of the states ϕ_i is not mapped to SAT
  - $\Rightarrow$ w is not synchronising
- $\Rightarrow$ w describes a satisfying assignment α for Φ
- $\Rightarrow \Phi \in$ 3-SAT

□

NP-Completeness

Proof.

- Function $f(\Phi)$ s.t. $\Phi \in$ 3-SAT $\Leftrightarrow f(\Phi) \in L_{SRW}$
- f computable in polynomial time
- $\Rightarrow L_{SRW}$ NP-hard
- $L_{SRW} \in$ NP $\Rightarrow L_{SRW}$ NP-complete

$\square$

Remarks

Lemma

$L_{ShortResetWord}$ *is NP-complete.*

Remark

$L_{ShortestResetWord}$ is NP-hard and coNP-hard [Olschewski and Ummels, 2010] $\Rightarrow L_{ShortestResetWord} \notin NP$, unless NP = coNP.

Černý conjecture

$$W_{sync}(\mathcal{A}) := \text{synchronising words of } \mathcal{A}$$

$$C(n) := \max\{ \min_{w \in W_{sync}(\mathcal{A})} (|w|) \mid \mathcal{A} \text{ is a DFA with } n \text{ states } \}$$

Lemma

$$C(n) \geq (n-1)^2$$

[Černý, 1964]

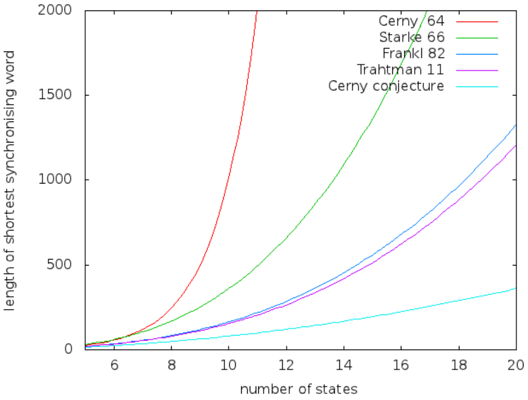
Černý conjecture

$$C(n) = (n-1)^2$$

Proven for several subsets of automata

Known results

- First polynomial bound: $C(n) \leq 1 + \frac{n(n-1)(n-2)}{2}$ [Starke, 1966]
- Simple bound in $\mathcal{O}(n^3)$: $\frac{n^3-n}{6}$ [Pin, 1983] and [Frankl, 1982]
- Recent improvement by $\frac{1}{8}$: $\frac{n(7n^2+6n-16)}{48}$ [Trahtman, 2011]



Conclusion

- Synchronising DFA: DFA with simple extra property
- Has applications in several other fields
- Check if given automaton is synchronising: polynomial time
- Check if given automaton has synchronising word of given length: NP-complete
- Question about shortest synchronising word for given number of states: Still open, possibly in $\mathcal{O}(n^2)$

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