Synchronizing Automata and the Černý Conjecture

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Seminar on Automata Theory

DFA

Definition

A DFA is a 5-tupel $(Q, \Sigma, \delta, q_0, F)$

with

- Q a finite set of states
- Σ an alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function
- $q_0 \in Q$ the starting state
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Synchronising DFA

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A DFA is a 3-tupel $(Q, \Sigma, \delta : \mathcal{P}(Q) \times \Sigma^* \to \mathcal{P}(Q))$.

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A word is *synchronising* with respect to an automaton $\mathcal{A} = (Q, \Sigma, \delta)$ if it leaves the automaton in a certain state, no matter which state the automaton started in.

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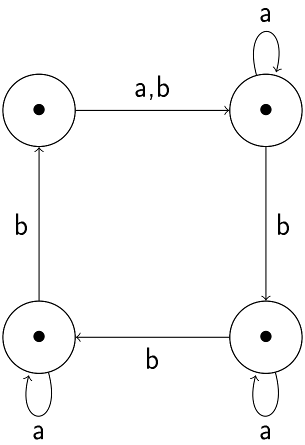
Definition

A *synchronising* DFA is a DFA that has a synchronising word.

Example



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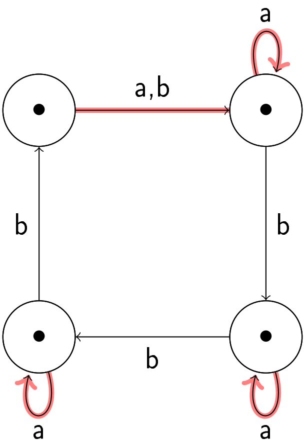

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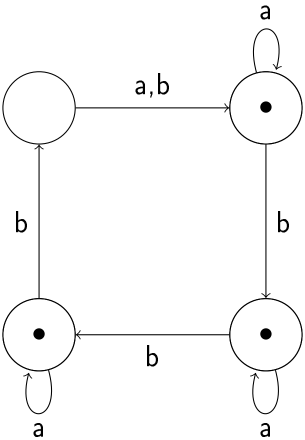

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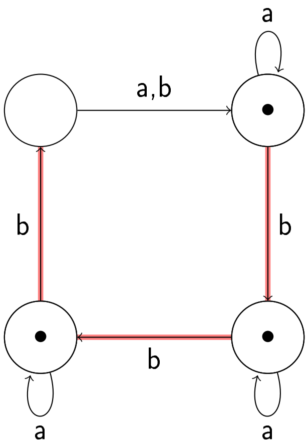

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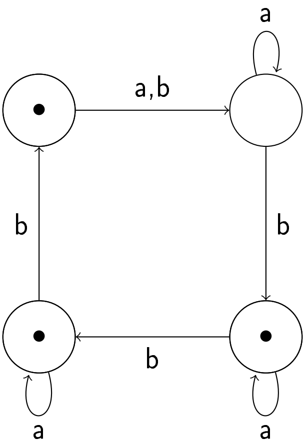

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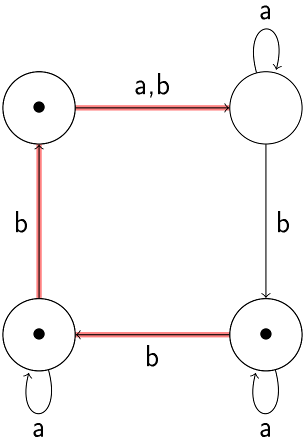

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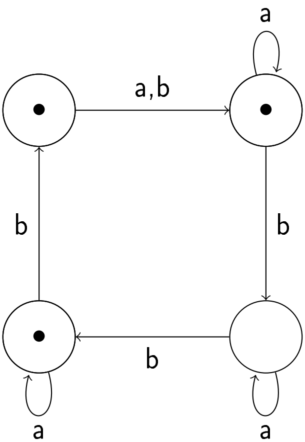

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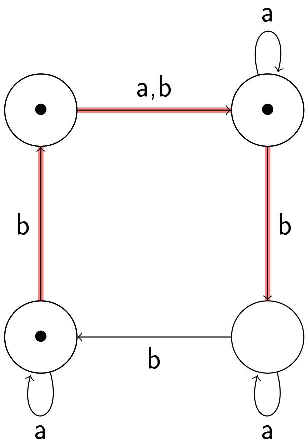

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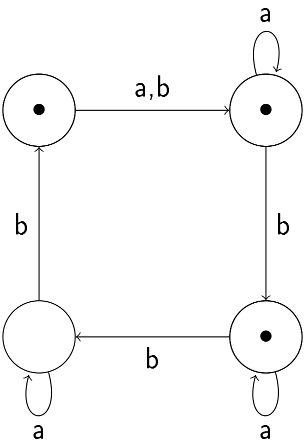

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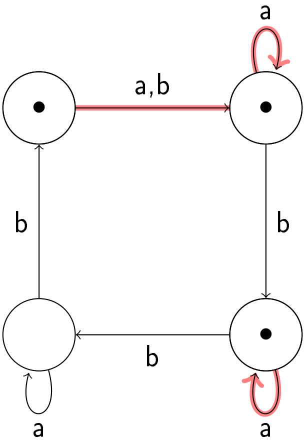

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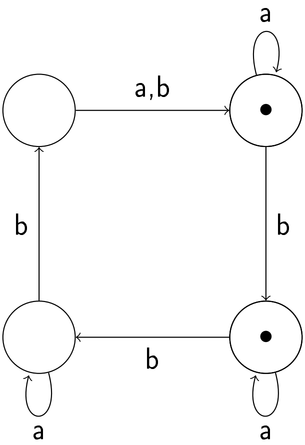

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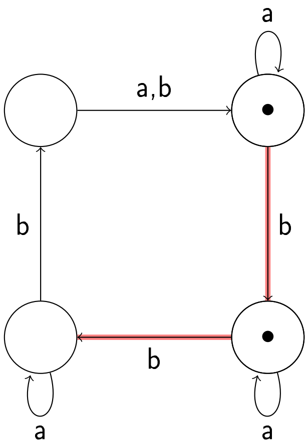

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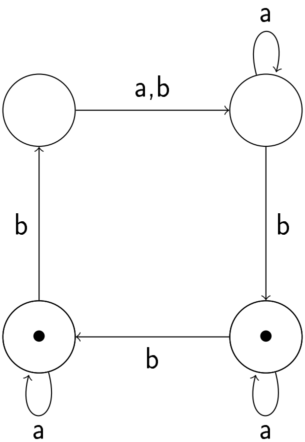

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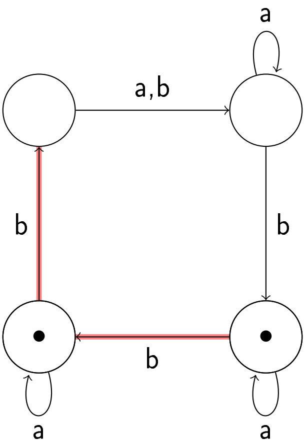

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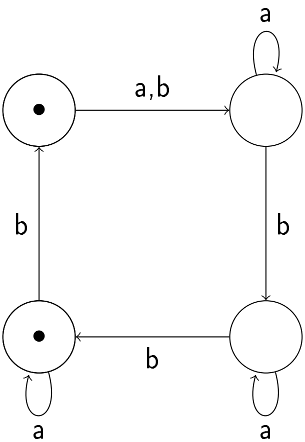

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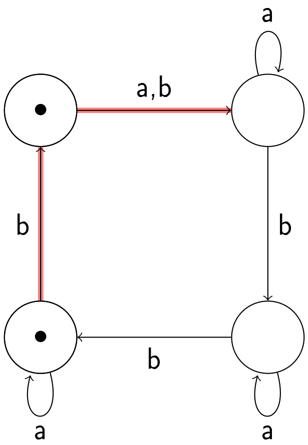

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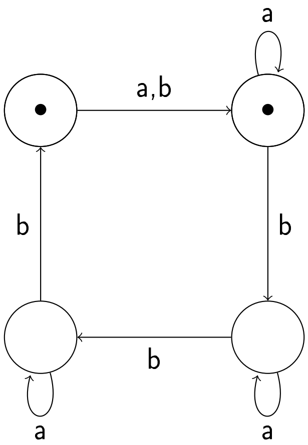

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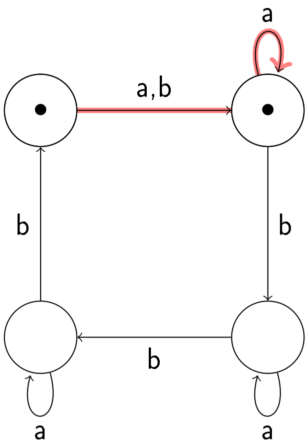

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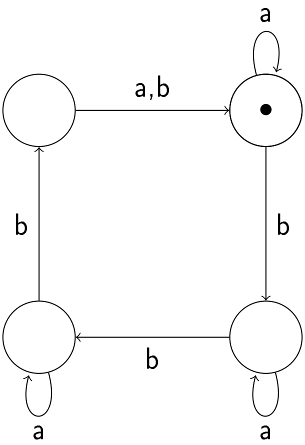

Extension

Observation

w is synchronising $\Rightarrow u \cdot w \cdot v$ is synchronising for all $u, v \in \Sigma^*$

Orientation on Conveyor Belts

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  - High production costs
- Better solution: Synchronising automata

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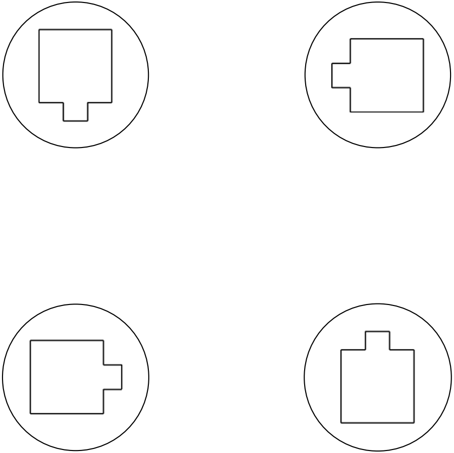

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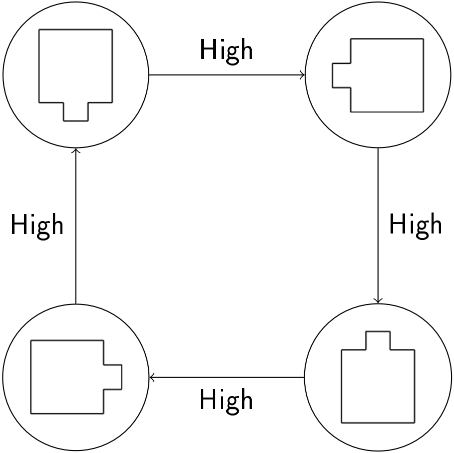

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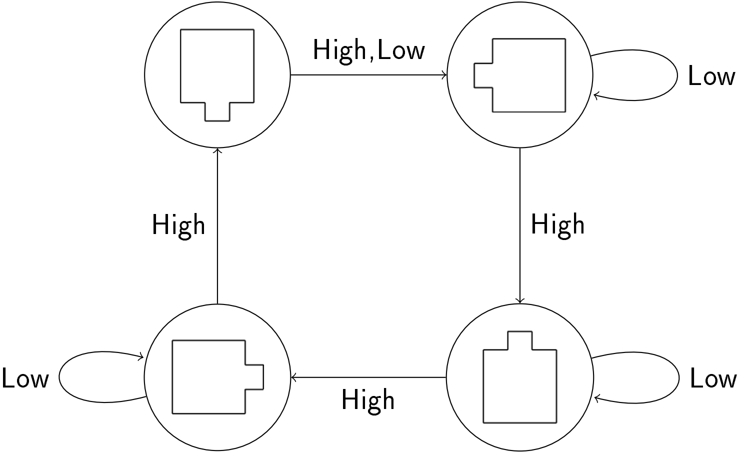

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Biological Automata

- [Benenson et al., 2003] presents biological DFA
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- Possibility for highly parallel computing
- Problem: Bring automata back to starting state
- Solution: Build synchronising automata

Checking for synchronising words

- $L_{sync} := \{\mathcal{A} \mid \mathcal{A} \text{ is a synchronising DFA}\}$
- Decision problem: Input DFA $\mathcal{A}$, Output $\mathcal{A} \in L_{sync}$

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- Idea: Keep track of states we can possibly be in

$\Rightarrow$ Power automaton

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- $\Rightarrow$ Singleton state reachable from Q in power automaton
- $\Rightarrow$ $\mathcal{A}$ is synchronising

□

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- New idea: Check reachability of singleton state from all $P \in \mathcal{P}(Q)$ with $|P| = 2$

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No information about length of shortest synchronising word anymore

Length of synchronising words

$$L_{ShortResetWord} := \{(\mathcal{A}, l) \mid \text{DFA } \mathcal{A} \text{ has synchronising word of length } l\}$$

$$L_{ShortestResetWord} := \{(\mathcal{A}, l) \mid \text{ Minimal synchronising word of DFA } \mathcal{A} \text{ has length } l\}$$

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Lemma

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3. $\Sigma := \{x, \neg x \mid x \in Var(\Phi)\}$

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4. $\delta(q, a) := \begin{cases} \text{SAT}, & q = \text{SAT} \\ & \\ & \end{cases}$

Reduction from 3-SAT

Input: Formula $\Phi = \bigwedge_{i=1}^{n} \phi_i$, $\phi_i = \bigvee_{j=1}^{3} x_{ij}$

1. $Var(\Phi)$:= Variables in Φ, $Lit(\phi)$:= Literals in ϕ
   - $\phi := (x_1, \neg x_1, x_2) \Rightarrow Var(\phi) = \{x_1, x_2\}, Lit(\phi) = \{x_1, \neg x_1, x_2\}$
2. $Q := \{\phi_i \mid 1 \leq i \leq n\} \cup Var(\Phi) \cup \{\text{SAT}\}$
3. $\Sigma := \{x, \neg x \mid x \in Var(\Phi)\}$
4. $\delta(q, a) := \begin{cases} \text{SAT}, & q \text{ represents clause } \bigvee_{i=1}^{3} x_i \text{ and } a = x_i \\ q, & q \text{ represents clause } \bigvee_{i=1}^{3} x_i \text{ and } a \neq x_i \\ & \\ & \end{cases}$

Reduction from 3-SAT

Input: Formula $\Phi = \bigwedge_{i=1}^{n} \phi_i$, $\phi_i = \bigvee_{j=1}^{3} x_{ij}$

1. $Var(\Phi) :=$ Variables in Φ, $Lit(\phi) :=$ Literals in ϕ
   - $\phi := (x_1, \neg x_1, x_2) \Rightarrow Var(\phi) = \{x_1, x_2\}, Lit(\phi) = \{x_1, \neg x_1, x_2\}$
2. $Q := \{\phi_i \mid 1 \leq i \leq n\} \cup Var(\Phi) \cup \{\text{SAT}\}$
3. $\Sigma := \{x, \neg x \mid x \in Var(\Phi)\}$
4. $$\delta(q, a) := \begin{cases} \\ \\ \\ \text{SAT}, & q \in Var(\Phi), a = q \text{ or } a = \neg q \\ q, & q \in Var(\Phi), a \neq q \text{ and } a \neq \neg q \end{cases}$$

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Output: Automaton $\mathcal{A} = (Q, \Sigma, \delta)$, $(\mathcal{A}, \qquad) \in L_{SRW} \Leftrightarrow \Phi \in 3-SAT$

Reduction from 3-SAT

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Output: Automaton $\mathcal{A} = (Q, \Sigma, \delta)$, $(\mathcal{A}, |Var(\Phi)|) \in L_{SRW} \Leftrightarrow \Phi \in 3-SAT$

Example

$$\Phi = (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_1 \vee x_4) \wedge (\neg x_3 \vee x_1 \vee \neg x_2)$$
$$\phi_1 = (x_1 \vee \neg x_3 \vee x_4), \phi_2 = (x_2 \vee \neg x_1 \vee x_4), \phi_3 = (\neg x_3 \vee x_1 \vee \neg x_2)$$

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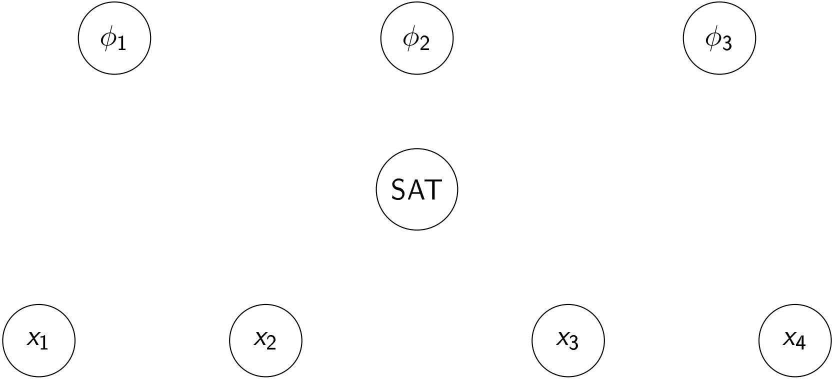
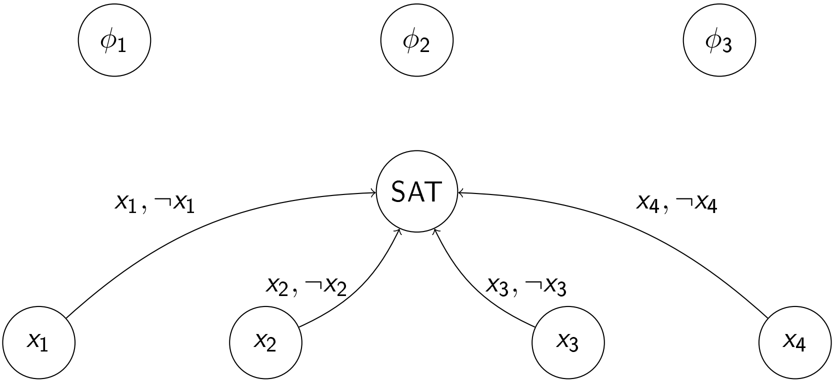
$$\Phi = (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_1 \vee x_4) \wedge (\neg x_3 \vee x_1 \vee \neg x_2)$$

$$\phi_1 = (x_1 \vee \neg x_3 \vee x_4), \phi_2 = (x_2 \vee \neg x_1 \vee x_4), \phi_3 = (\neg x_3 \vee x_1 \vee \neg x_2)$$



Example

$$\Phi = (x_1 \lor \neg x_3 \lor x_4) \land (x_2 \lor \neg x_1 \lor x_4) \land (\neg x_3 \lor x_1 \lor \neg x_2)$$
$$\phi_1 = (x_1 \lor \neg x_3 \lor x_4), \phi_2 = (x_2 \lor \neg x_1 \lor x_4), \phi_3 = (\neg x_3 \lor x_1 \lor \neg x_2)$$



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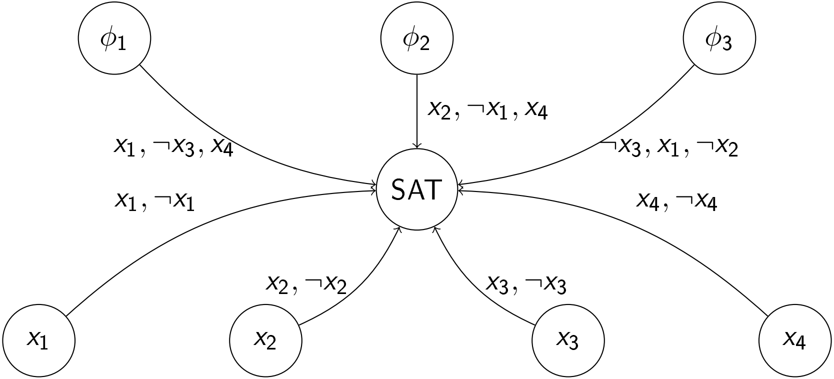

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$$\Phi = (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_1 \vee x_4) \wedge (\neg x_3 \vee x_1 \vee \neg x_2)$$
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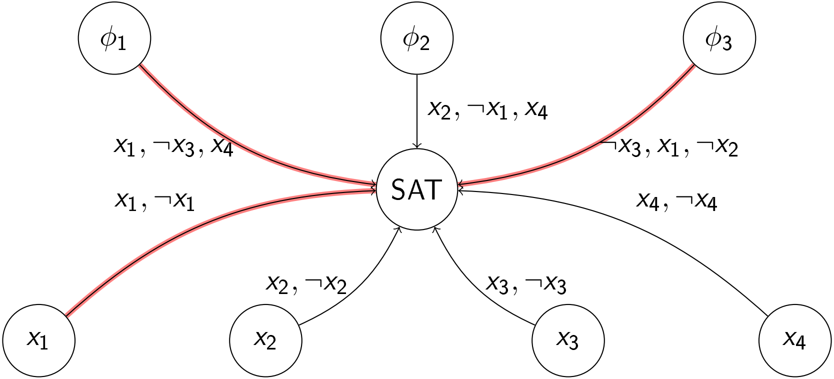
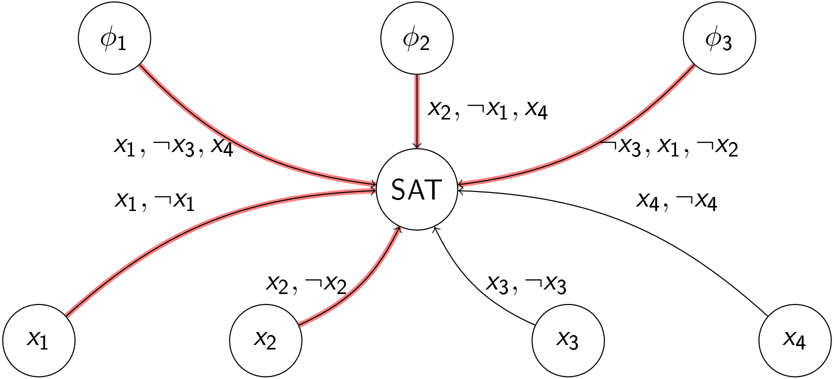


$$\alpha(x_1) = 1$$

Example

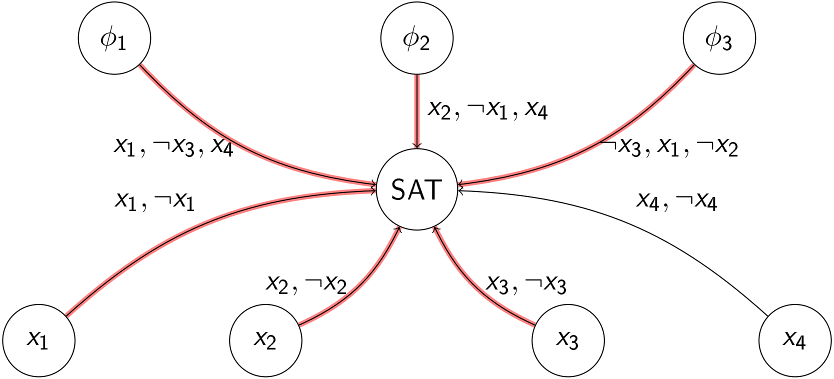
$$\Phi = (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_1 \vee x_4) \wedge (\neg x_3 \vee x_1 \vee \neg x_2)$$
$$\phi_1 = (x_1 \vee \neg x_3 \vee x_4), \phi_2 = (x_2 \vee \neg x_1 \vee x_4), \phi_3 = (\neg x_3 \vee x_1 \vee \neg x_2)$$



$$\alpha(x_1) = 1 \ \alpha(x_2) = 1$$

Example

$$\Phi = (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_1 \vee x_4) \wedge (\neg x_3 \vee x_1 \vee \neg x_2)$$
$$\phi_1 = (x_1 \vee \neg x_3 \vee x_4), \phi_2 = (x_2 \vee \neg x_1 \vee x_4), \phi_3 = (\neg x_3 \vee x_1 \vee \neg x_2)$$

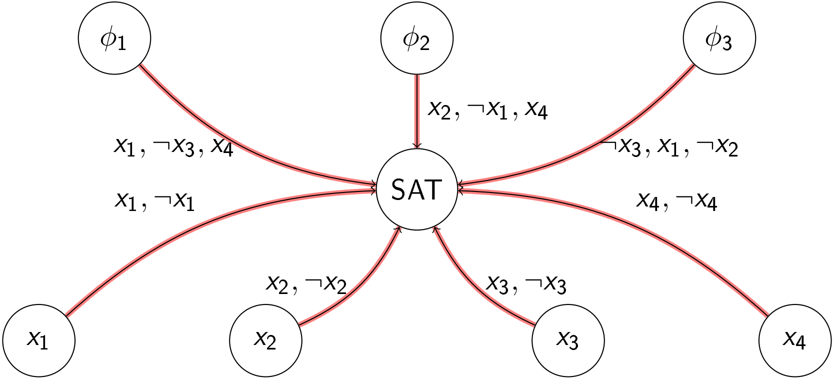


$$\alpha(x_1) = 1 \;\alpha(x_2) = 1 \;\alpha(x_3) = 0$$

Example

$$\Phi = (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_1 \vee x_4) \wedge (\neg x_3 \vee x_1 \vee \neg x_2)$$

$$\phi_1 = (x_1 \vee \neg x_3 \vee x_4), \phi_2 = (x_2 \vee \neg x_1 \vee x_4), \phi_3 = (\neg x_3 \vee x_1 \vee \neg x_2)$$



$$\alpha(x_1) = 1 \;\; \alpha(x_2) = 1 \;\; \alpha(x_3) = 0 \;\; \alpha(x_4) = 0$$

Runtime

States	$\lvert\{\phi_i\}\rvert + \lvert\{Var(\Phi)\}\rvert + 1$

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$2 \cdot \lvert Var(\Phi) \rvert$	

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$\mathcal{O}(3n)$	

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$\mathcal{O}(3n)$	$\mathcal{O}(n)$

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$\mathcal{O}(3n)$	$\mathcal{O}(n)$
Transitions	$\lvert Q\rvert \cdot \lvert\Sigma\rvert$	

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$\mathcal{O}(3n)$	$\mathcal{O}(n)$
Transitions	$\mathcal{O}(n) \cdot \mathcal{O}(n)$	

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$\mathcal{O}(3n)$	$\mathcal{O}(n)$
Transitions	$\mathcal{O}(n) \cdot \mathcal{O}(n)$	$\mathcal{O}(n^2)$

Runtime

States	$\mathcal{O}(n) + \mathcal{O}(3n)$	$\mathcal{O}(n)$
Alphabet	$\mathcal{O}(3n)$	$\mathcal{O}(n)$
Transitions	$\mathcal{O}(n) \cdot \mathcal{O}(n)$	$\mathcal{O}(n^2)$
Accumulated	$\mathcal{O}(n^2)$	

Correctness

$\Phi \in 3 - SAT \Rightarrow (\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$.

- $\Phi = \bigwedge_{i=1}^{n} \phi_i \in 3 - SAT$
- $\Rightarrow$ There is an assignment α making at least one literal in every clause true
- Pick $w := a_1 \ldots a_n$ with $a_i = \begin{cases} x_i, & \alpha(x_i) = 1 \\ \neg x_i, & \text{else} \end{cases}$

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- Pick $w := a_1 \ldots a_n$ with $a_i = \begin{cases} x_i, & \alpha(x_i) = 1 \\ \neg x_i, & \text{else} \end{cases}$
    - w maps every state ϕ_i to SAT
    - w maps every literal x_i to SAT
    - $\Rightarrow$ w is synchronising

Correctness

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- $\Phi = \bigwedge_{i=1}^{n} \phi_i \in 3 - SAT$
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- Pick $w := a_1 \ldots a_n$ with $a_i = \begin{cases} x_i, & \alpha(x_i) = 1 \\ \neg x_i, & \text{else} \end{cases}$
  - w maps every state ϕ_i to SAT
  - w maps every literal x_i to SAT
  - $\Rightarrow w$ is synchronising
- $\Rightarrow \mathcal{A}$ has a synchronising word of length $|Var(\Phi)|$
- $\Rightarrow (\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$

□

Correctness (cont.)

$(\mathcal{A}, |Var(\Phi)|) \in L_{SRW} \Rightarrow \Phi \in 3-SAT.$

- $(\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$
- $\Rightarrow \mathcal{A}$ has a synchronising word w of length $|Var(\Phi)|$

Correctness (cont.)

$(\mathcal{A}, |Var(\Phi)|) \in L_{SRW} \Rightarrow \Phi \in 3-SAT$.

- $(\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$
- $\Rightarrow \mathcal{A}$ has a synchronising word w of length $|Var(\Phi)|$
  - w does not describe a valid assignment $\Rightarrow$ w is not synchronising
  - $\Rightarrow$ w describes a valid assignment

Correctness (cont.)

$(\mathcal{A}, |Var(\Phi)|) \in L_{SRW} \Rightarrow \Phi \in 3-SAT.$

- $(\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$
- $\Rightarrow \mathcal{A}$ has a synchronising word w of length $|Var(\Phi)|$
  - w does not describe a valid assignment $\Rightarrow$ w is not synchronising
  - $\Rightarrow$ w describes a valid assignment

  - Implied assignment α does not satisfy Φ
  - $\Rightarrow$ one of the states ϕ_i is not mapped to SAT
  - $\Rightarrow$ w is not synchronising

Correctness (cont.)

$(\mathcal{A}, |Var(\Phi)|) \in L_{SRW} \Rightarrow \Phi \in 3-SAT$.

- $(\mathcal{A}, |Var(\Phi)|) \in L_{SRW}$
- $\Rightarrow \mathcal{A}$ has a synchronising word w of length $|Var(\Phi)|$
  - w does not describe a valid assignment $\Rightarrow$ w is not synchronising
  - $\Rightarrow$ w describes a valid assignment

  - Implied assignment α does not satisfy Φ
  - $\Rightarrow$ one of the states ϕ_i is not mapped to SAT
  - $\Rightarrow$ w is not synchronising
- $\Rightarrow$ w describes a satisfying assignment α for Φ
- $\Rightarrow \Phi \in$ 3-SAT

□

NP-Completeness

Proof.

- Function $f(\Phi)$ s.t. $\Phi \in$ 3-SAT $\Leftrightarrow f(\Phi) \in L_{SRW}$
- f computable in polynomial time
- $\Rightarrow L_{SRW}$ NP-hard
- $L_{SRW} \in$ NP $\Rightarrow L_{SRW}$ NP-complete

$\square$

Remarks

Lemma

$L_{ShortResetWord}$ is NP-complete.

Remark

$L_{ShortestResetWord}$ is NP-hard and coNP-hard
[Olschewski and Ummels, 2010] $\Rightarrow L_{ShortestResetWord} \notin NP$, unless NP = coNP.

Černý conjecture

$$W_{sync}(\mathcal{A}) := \text{synchronising words of } \mathcal{A}$$

$$C(n) := \max\{ \min_{w \in W_{sync}(\mathcal{A})} (|w|) \mid \mathcal{A} \text{ is a DFA with } n \text{ states } \}$$

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Lemma

$$C(n) \geq (n-1)^2$$

[Černý, 1964]

Černý conjecture

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Lemma

$$C(n) \geq (n-1)^2$$

[Černý, 1964]

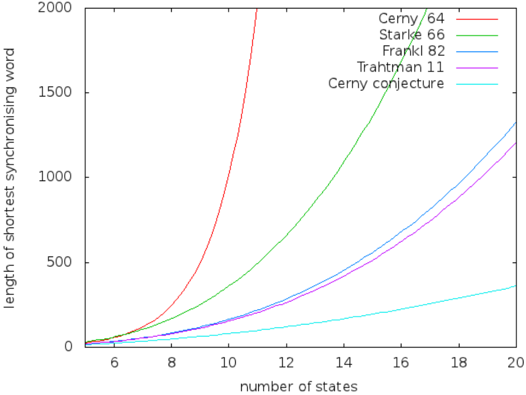
Černý conjecture

$$C(n) = (n-1)^2$$

Proven for several subsets of automata

Known results

- First polynomial bound: $C(n) \leq 1 + \frac{n(n-1)(n-2)}{2}$ [Starke, 1966]
- Simple bound in $\mathcal{O}(n^3)$: $\frac{n^3-n}{6}$ [Pin, 1983] and [Frankl, 1982]
- Recent improvement by $\frac{1}{8}$: $\frac{n(7n^2+6n-16)}{48}$ [Trahtman, 2011]



Conclusion

- Synchronising DFA: DFA with simple extra property
- Has applications in several other fields
- Check if given automaton is synchronising: polynomial time
- Check if given automaton has synchronising word of given length: NP-complete
- Question about shortest synchronising word for given number of states: Still open, possibly in $\mathcal{O}(n^2)$

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