Easy to Win, Hard to Master:
Optimal Strategies in Parity Games with Costs

Joint work with Martin Zimmermann

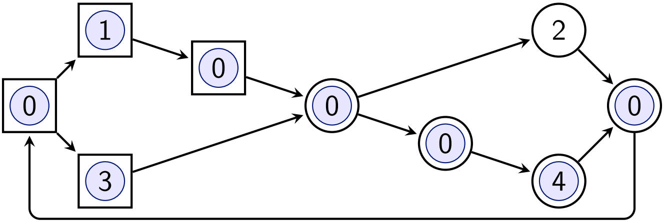
Alexander Weinert

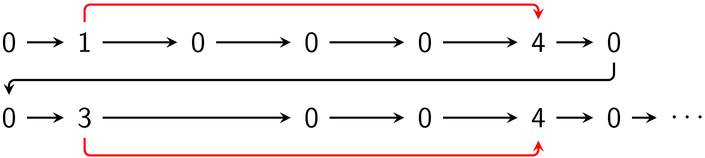
Saarland University

August 31st, 2016

CSL 2016 - Marseille, France

Parity Games

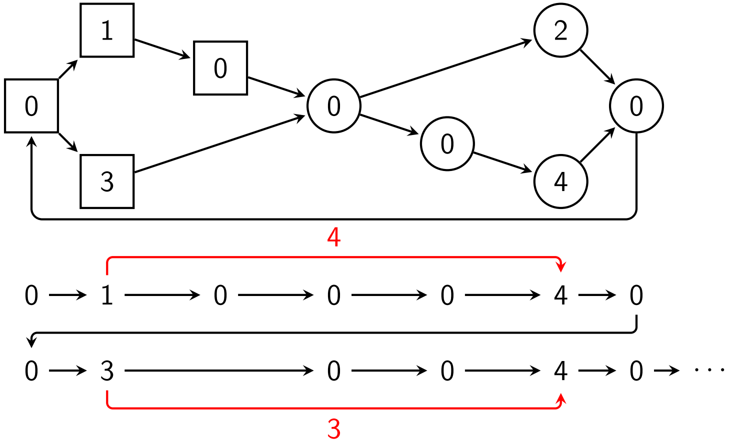




Deciding winner in $\mathrm{UP} \cap \textsc{co}\text{-}\mathrm{UP}$ Positional Strategies

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Finitary Parity Games



Goal for Player 0: Bound response times

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Decision Problem

Theorem (Chatterjee et al., Finitary Winning, 2009)

The following decision problem is in PTime*:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \text{FinParity}(\Omega))$

Question: *Does there exist a strategy* σ *with* $\text{Cst}(\sigma) < \infty$*?*

Theorem

The following decision problem is PSpace*-complete:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \text{FinParity}(\Omega))$*,*
bound $b \in \mathbb{N}$

Question: *Does there exist a strategy* σ *with* $\text{Cst}(\sigma) \leq b$*?*

Introduction ✓

Complexity

in PSpace

PSpace-hard

Exponential Memory

Sufficient

Necessary

From Finitary Parity to Parity

Given: Finitary parity game $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$, bound $b \in \mathbb{N}$.

Lemma

Deciding if Player 0 has strategy σ with $\mathrm{Cst}(\sigma) \leq b$ is in $\mathrm{PS}\textsc{pace}$.

Idea: Simulate game, keeping track of open requests.

Lemma

Player 0 has such a strategy iff she "survives" $p(|\mathcal{G}|)$ steps in extended game $\mathcal{G}'$.

Algorithm:

Simulate all plays in $\mathcal{G}'$ on-the-fly for $p(|\mathcal{G}|)$ steps using an alternating Turing machine.

$\Rightarrow$ Problem is in $\mathrm{APT}\textsc{ime}$

(Chandra et al., Alternation, 1981)

$\Rightarrow$ Problem is in $\mathrm{PS}\textsc{pace}$

Introduction ✓

Complexity

in PSpace ✓

PSpace-hard

Exponential Memory

Sufficient

Necessary

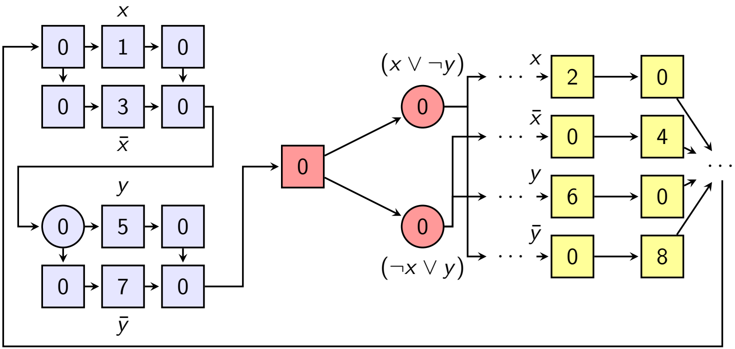
PSPACE-completeness

Lemma

The given decision problem is PSPACE*-hard.*

Idea: Reduction from Quantified Boolean Formulas, e.g.:

$$\forall x\ \exists y\ .\ (x \lor \neg y) \land (\neg x \lor y)$$



Introduction ✓

Complexity

in PSpace ✓

PSpace-hard ✓

Exponential Memory

Sufficient

Necessary

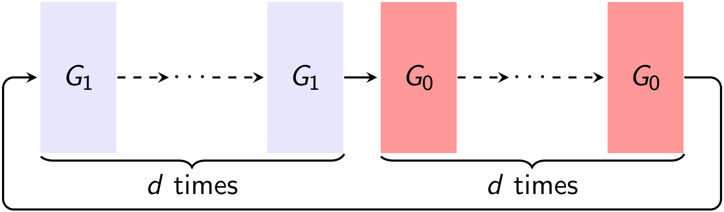
Memory Requirements (for Player 0)

Theorem

Optimal strategies for parity games require exponential memory.

Sufficiency: Corollary of proof of $\textsc{PSpace}$-membership

Necessity: Construct family $\mathcal{G}_d$:



(Fijalkow and Chatterjee, Infinite-state games, 2013)

Player 0 needs to store d choices of d possible values each
$\Rightarrow$ Player 0 requires $\approx 2^d$ many memory states

Introduction ✓

Complexity

in PSpace ✓

PSpace-hard ✓

Exponential Memory

Sufficient ✓

Necessary ✓

Conclusion

	Parity	Finitary Parity	
		Winning	Optimal
Complexity	$\mathrm{UP} \cap \text{co-}\mathrm{UP}$	PTime	PSpace-comp.
Strategies	1	1	Exp.

Take-away: Forcing Player 0 to answer quickly in (finitary) parity games makes it harder

- to decide whether she can satisfy the bound
- for her to play the game

Introduction ✓

Complexity

in PSpace ✓

PSpace-hard ✓

Exponential Memory

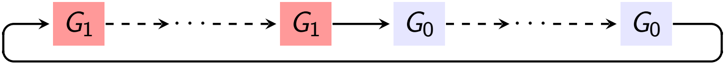
Sufficient ✓

Necessary ✓

Tradeoffs

Parity Games with Costs

Tradeoffs



	Winning				Optimal
Size	1	d	$\cdots$	2^{d-1}	2^d
Cost	$3d$	$3d-1$	$\cdots$	$2d+1$	$2d$

Introduction ✓

Complexity

in PSpace ✓

PSpace-hard ✓

Exponential Memory

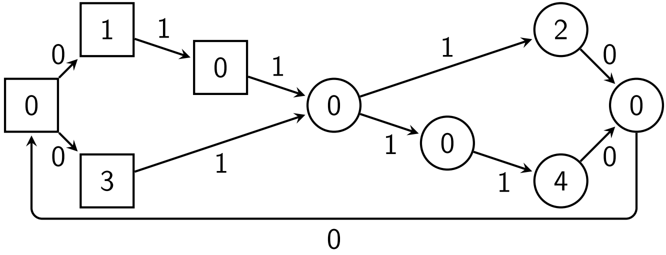
Sufficient ✓

Necessary ✓

Tradeoffs ✓

Parity Games with Costs

Parity Games with Cost



Finitary parity games are special case

$\Rightarrow$ $\mathrm{PS{\small PACE}}$-hard $\Rightarrow$ Exp. memory necessary

Algorithm for solving finitary games works as well

$\Rightarrow$ In $\mathrm{PS{\small PACE}}$ $\Rightarrow$ Exp. memory sufficient

Conclusion

	Parity	Cost-Parity	
		Winning	Optimal
Complexity	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$	$\mathrm{PS}{\small\mathrm{PACE}}$-comp.
Strategies	1	1	Exp.

Take-away: Forcing Player 0 to answer quickly in parity games makes it harder

- to decide whether she can satisfy the bound
- for her to play the game