Easy to Win, Hard to Master:
Optimal Strategies in Parity Games with Costs

Joint work with Martin Zimmermann

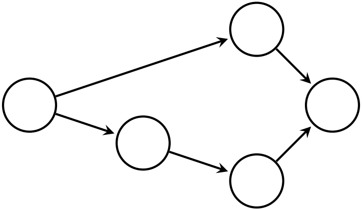
Alexander Weinert

Saarland University

September 9th, 2016

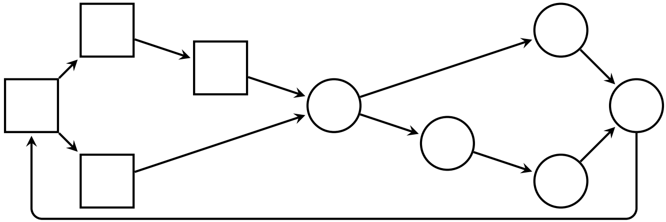
Highlights 2016 - Brussels

Parity Games



Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games



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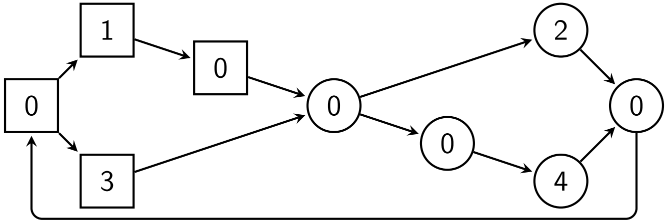

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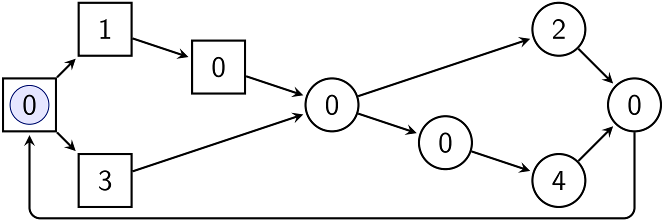

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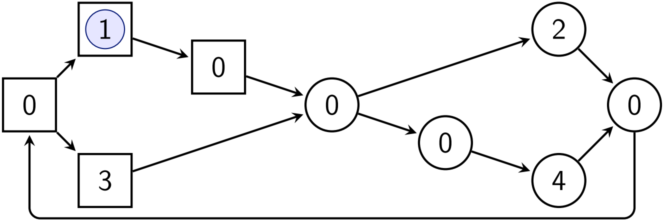
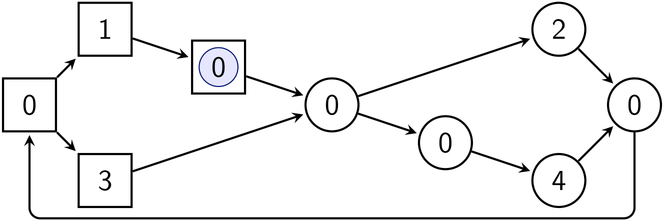


$0 \longrightarrow 1$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games



$0 \longrightarrow 1 \longrightarrow 0$

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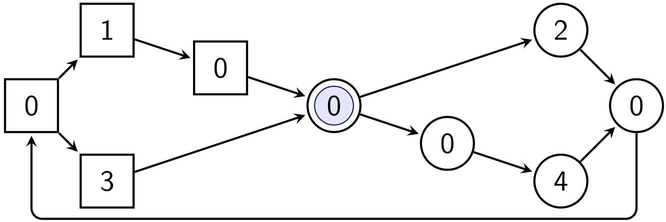

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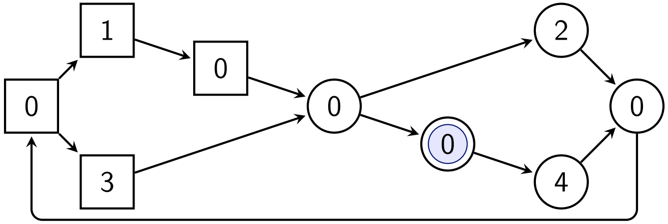

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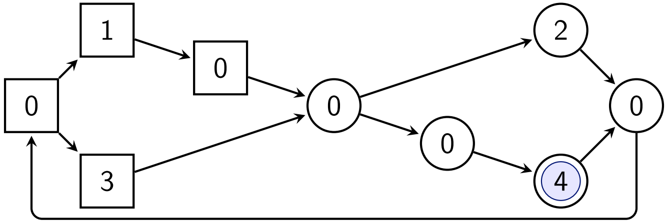

$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4$

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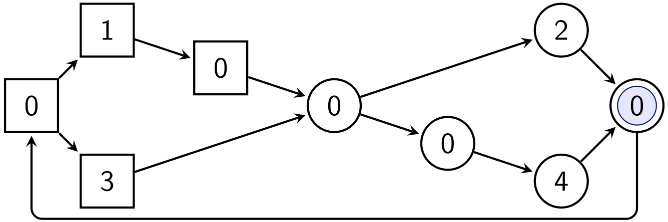

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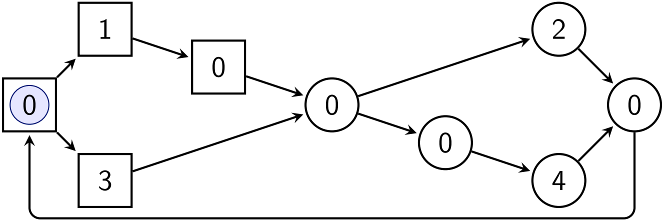

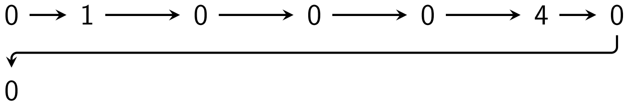

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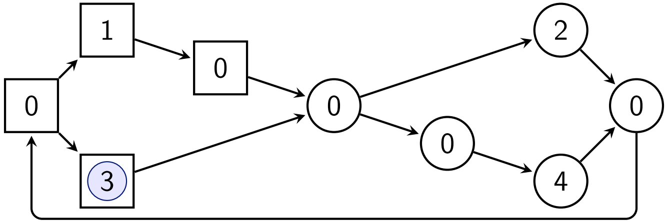

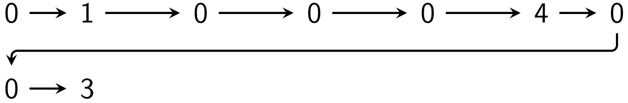

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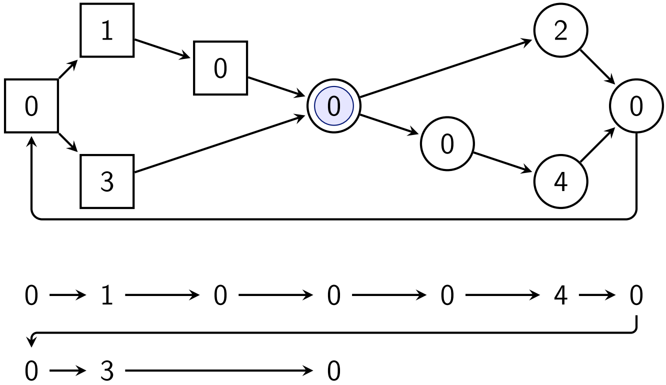

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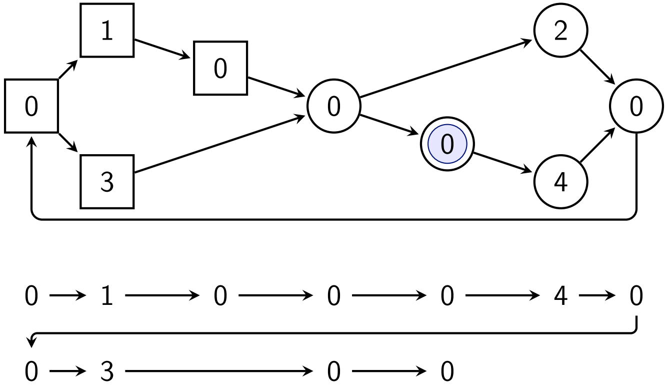

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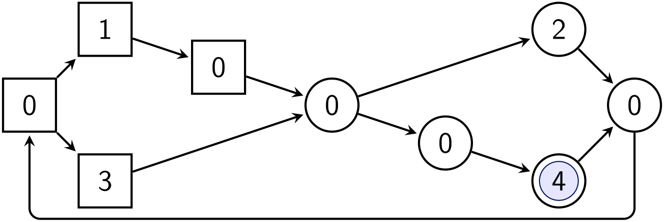

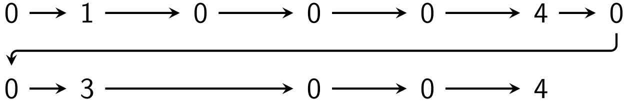

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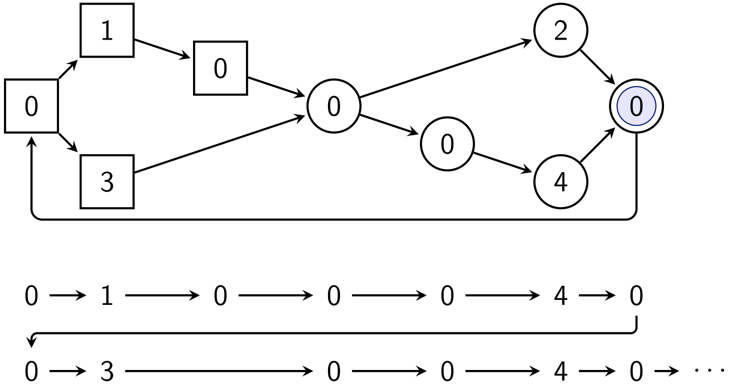

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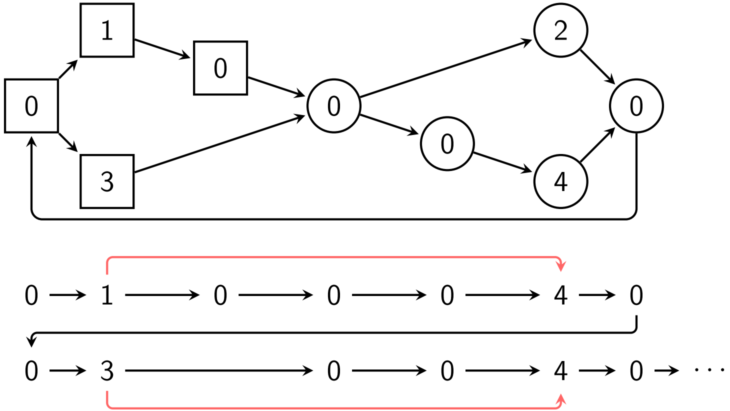

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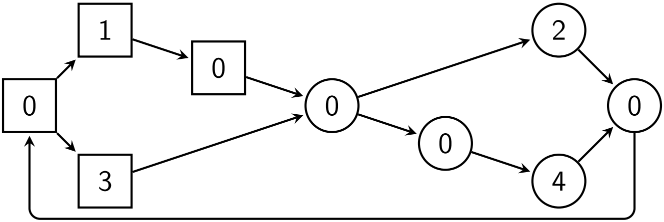
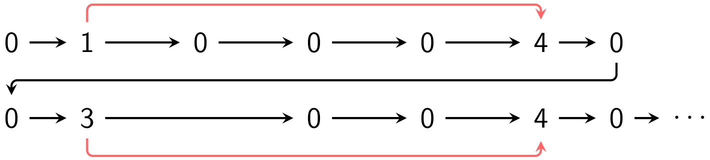


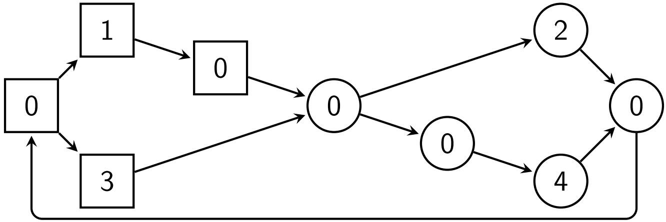


Deciding winner in $\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$

Positional Strategies

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Finitary Parity Games



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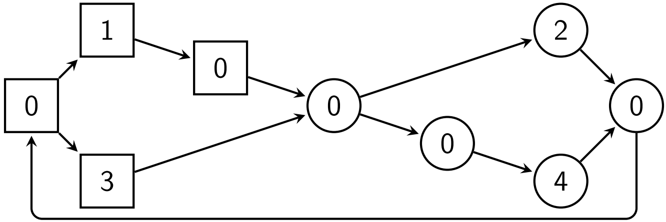

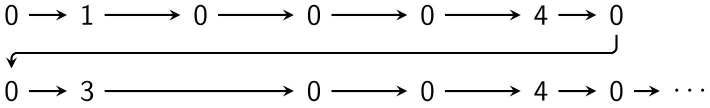

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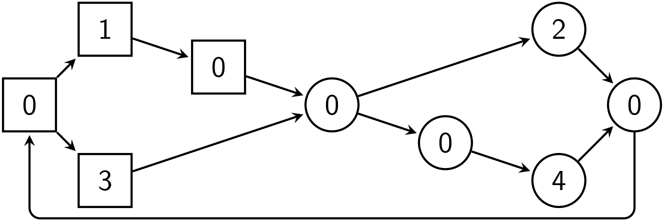

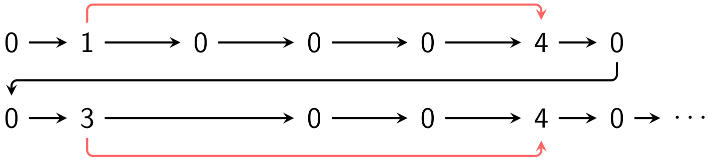

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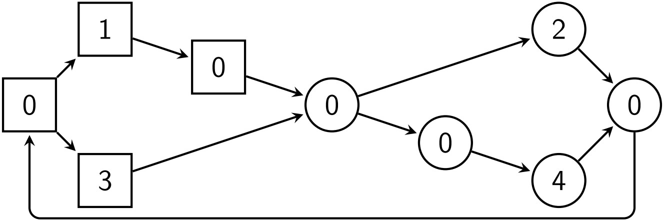

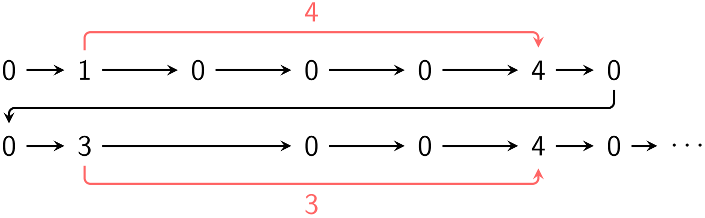

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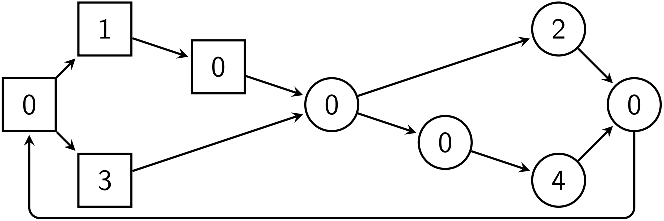

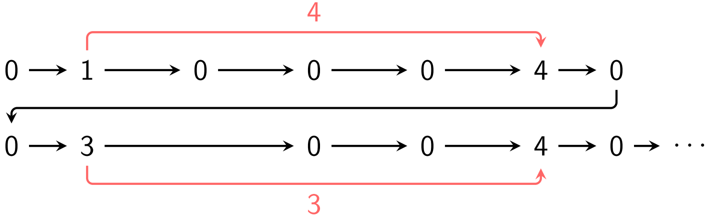

Goal for Player 0: Bound response times

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Decision Problem

Theorem (Chatterjee et al., Finitary Winning, 2009)

The following decision problem is in $\textsc{PTime}$*:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$

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The following decision problem is $\textsc{PSpace}$*-complete:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$, *bound* $b \in \mathbb{N}$

Question: *Does there exist a strategy* σ *with* $\mathrm{Cst}(\sigma) \leq b$*?*

From Finitary Parity to Parity

Given: Finitary parity game $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$, bound $b \in \mathbb{N}$.

Lemma

Deciding if Player 0 has strategy σ with $\mathrm{Cst}(\sigma) \leq b$ is in $\mathrm{PS}\textsc{pace}$.

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$\Rightarrow$ Problem is in $\mathrm{APT}{\small \mathrm{IME}}$

(Chandra et al., Alternation, 1981)

$\Rightarrow$ Problem is in $\mathrm{PS}{\small \mathrm{PACE}}$

PSpace-completeness

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The given decision problem is in PSPACE.

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$\Rightarrow$ The given decision problem is PSPACE-complete

Memory Requirements (for Player 0)

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Optimal strategies for finitary parity games need exponential memory

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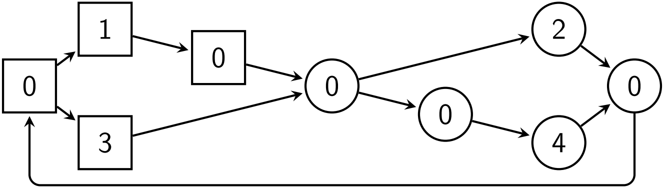

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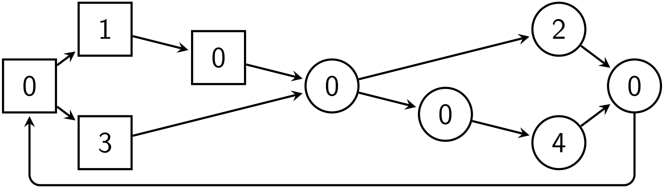

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- Generalize to d colors
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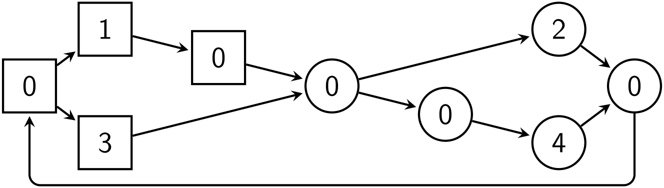

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Player 1 has d choices of d actions

$\Rightarrow$

Player 0 needs $\approx 2^d$ memory states

Conclusion

	Parity	
Complexity	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$	
Strategies	1	

Conclusion

	Parity	Finitary Parity
		Winning
Complexity	UP ∩ co-UP	PTime
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Complexity	UP ∩ co-UP	PTime	PSpace-comp.
Strategies	1	1	Exp.

Take-away: Forcing Player 0 to answer quickly in (finitary) parity games makes it harder

- to decide whether she can satisfy the bound
- for her to play the game